Application of Yu's Variational Method to Heat Conduction of Solid with Phase Change.

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By refering to Yu's variational method, a sufficiently long melting slab is investigated. The slab is acted upon by a prescribed heat input at one face and has its other face insulated. In order to find a solution involving two unknown functions, the heat balance integral method introduced by Goodman is used as a subsidary condition.

§ 1. Introduction

Yu and Vujanovic derived the variational formulation of heat conduction of rod introducing the variational invariant[1),2)]

$$V = \int_0^L \left\{ c\frac{\partial \theta_1}{\partial t}\theta + \frac{\lambda}{2}\left(\frac{\partial \theta}{\partial x}\right)^2 \right\} dx, \qquad (1.1)$$

where c is the heat capacity per unit volume, θ the temperature change, λ the heat conductivity, L the length of the rod.

The suffix O denotes the quantity not subjected to any variation, therefore it becomes $\theta = \theta_0$ after the variational process.

We shall examine to evaluate the problem of moving boundary, that is the heat conduction of solid with phase change, by applying their theory. We also use the heat balance integral method proposed by Goodman[3),4)] as a subsidary condition.

§ 2. Basic Formulation

Take a sufficiently thick slab of thickness L, occupying the region $(0, L)$, insulated at $x = L$, exposed to a prescribed heat input $Q(t)$ at $x = 0$. It will be assumed here that the melted portion is immediately removed. Let $s = s(t)$ denote the thickness of the portion of the material which has melted.

We introduce the variational invariant

$$V = \int_s^L \left\{ c\frac{\partial \theta_0}{\partial t}\theta + \frac{\lambda}{2}\left(\frac{\partial \theta}{\partial x}\right)^2 \right\} dx, \qquad (2.1)$$

and take the variations as the changes of the quantities due to the virtual displacement of the position of the melting line $s(t)$. The variation of V is evaluated as

$$\delta V = \left\{ -c_m \left(\frac{\partial \theta_0}{\partial t}\right)_m \theta_m - \frac{\lambda_m}{2}\left(\frac{\partial \theta}{\partial x}\right)_m^2 \right\}\delta s + \int_s^L \left\{ c\frac{\partial \theta_0}{\partial t}\,\delta\theta + \lambda\frac{\partial \theta}{\partial x}\frac{\partial}{\partial x}(\delta\theta) \right\} dx, \quad \cdots\cdots (2.2)$$

where suffix m denotes the melting state. Integrating by part and using the fact $(\delta\theta)_{\mathrm{m}}=0$, we see

$$\int_s^L \lambda\frac{\partial\theta}{\partial x}\frac{\partial}{\partial x}(\delta\theta)\,dx = -\int_s^L \frac{\partial}{\partial x}\left(\lambda\frac{\partial\theta}{\partial x}\right)\delta\theta dx. \quad \cdots\cdots (2.3)$$

Inserting eq. (2.3) into eq. (2.2) by considering the heat conduction equation, we see

$$\delta V = -\left\{ c_m\left(\frac{\partial\theta_0}{\partial t}\right)_m\theta_m + \frac{\lambda_m}{2}\left(\frac{\partial\theta}{\partial x}\right)_m^2\right\}\delta s. \quad \cdots\cdots (2.4)$$

This is the variational equation we found.

We shall try to find the solution of the following type, [1), 2), 5)]

$$\theta = \left(1-\frac{x}{L}\right)^2 f(t). \quad \cdots\cdots (2.5)$$

This solution has two parameters, $s(t)$ and $f(t)$. Therefore we must find the subsidary condition of eq. eq. (2.4). The heat balance integral method introduced by Godman[3), 4)] is chosen for this aim.

Introduce the quantity

$$I = \int_s^L c\theta dx, \quad \cdots\cdots (2.6)$$

and differentiate with respect to time by considering the heat conduction equation, we find the followings :

$$\frac{dI}{dt} = -c_m\dot{\theta}_m\dot{s} + \int_s^L c\dot{\theta}dx = -c_m\theta_m\dot{s} + \int_s^L\frac{\partial}{\partial x}\left(\lambda\frac{\partial\theta}{\partial x}\right)dx = -c_m\,\theta_m\dot{s} - \lambda_m\left(\frac{\partial\theta}{\partial x}\right)_m.$$

Inserting the boundary condition of the melting line[3), 4), 6)]

$$-\lambda_m\left(\frac{\partial\theta}{\partial x}\right)_m = Q(t) - \rho l\,\dot{s}, \quad \cdots\cdots (2.7)$$

where ρl ir the latent heat per unit volume, we have

$$\frac{dI}{dt} = -(c_m\theta_m + \rho l)\,\dot{s} + Q(t). \quad \cdots\cdots (2.8)$$

§ 3. Method of Solution

In this section, we shall find the solution of eq. (2.4) with the subsidary condition (2.8). We set the solution as eq. (2.5), also we set

$$\theta_0 = \left(1-\frac{x}{L}\right)^2 f_0(t). \quad \cdots\cdots (3.1)$$

Inserting eqs. (2.5) and (3.1) into eq. (2.1), we have

$$V = \frac{cL}{5}\left(1-\frac{s}{L}\right)^5\dot{f}_0 f + \frac{2\lambda}{3L}\left(1-\frac{s}{L}\right)^3 f^2. \quad \cdots\cdots (3.2)$$

Therefore, we see

[left side of eq. (2.4)] $= -\left[c\left(1-\frac{s}{L}\right)^4\dot{f}_0 f + \frac{2\lambda}{L^2}\left(1-\frac{s}{L}\right)^2 f^2\right]\delta s + \left[\frac{cL}{5}\left(1-\frac{s}{L}\right)^5\dot{f}_0\right.$

$$+\frac{4\lambda}{3L}\left(1-\frac{s}{L}\right)^3 f\Big]\,\delta f. \quad (3.3)$$

Also, using eqs. (2.5) and (3.1), we have

$$[\text{right side of eq. (2.4)}] = -\left[c_m\left(1-\frac{s}{L}\right)^2\theta_m\dot{f}_0(t)+\frac{2\lambda_m}{L^2}\left(1-\frac{s}{L}\right)^2\{f(t)\}^2\right]\delta s. \quad (3.4)$$

Equating eq. (3.3) and eq. (3.4) and seting $f_0 = f$, we find

$$\left\{c_m\theta_m\dot{f}+\frac{2}{L^2}(\lambda_m-\lambda)f^2-c\left(1-\frac{s}{L}\right)^2 f\dot{f}\right\}\delta s+\left\{\frac{cL}{5}\left(1-\frac{s}{L}\right)^3\dot{f}+\frac{4\lambda}{3L}\left(1-\frac{s}{L}\right)f\right\}\delta f=0. \quad (3.5)$$

Also, inserting eq. (2.5) into eq. (2.6), we have

$$I=\frac{cL}{3}\left(1-\frac{s}{L}\right)^3 f(t). \quad (3.6)$$

Let us set the origin of time as the time when the melting beings, i.e.

$$s(0)=0. \quad (3.7)$$

Integrating eq. (2.8) and substituting eq. (3.6), we have

$$\frac{cL}{3}\left\{\left(1-\frac{s}{L}\right)^3 f(t)-\theta_m\right\}=-(c_m\theta_m+\rho l)\,s+\int_0^t Q(t)\,dt. \quad (3.8)$$

Here, we set

$$f(0)=\theta_m \quad (3.9)$$

From eq. (3.8), the relation of δs and δf is found as

$$\delta f=\frac{3}{cL}\frac{1}{\left(1-\frac{s}{L}\right)^3}\left\{c\left(1-\frac{s}{L}\right)^2-(c_m\theta_m+\rho l)\right\}\delta s. \quad (3.10)$$

Eliminating δf and δs from eqs. (3.5) and (3.10), we have

$$\left\{c_m\theta_m\dot{f}+\frac{2}{L}(\lambda_m-\lambda)f^2-c\left(1-\frac{s}{L}\right)^2 f\dot{f}\right\}\left(1-\frac{s}{L}\right)^2-\frac{3}{cL}\left\{(c_m\theta_m+\rho l)-c\left(1-\frac{s}{L}\right)^2 f\right\}$$
$$\left\{\frac{cL}{5}\left(1-\frac{s}{L}\right)^2\dot{f}+\frac{4\lambda}{3L}f\right\}=0. \quad (3.11)$$

Here, we find the simultaneous equations (3.8) and (3.11).

For avoiding the troublesome calculations, we assume $c_m = c$ and $\lambda_m = \lambda$. Using Adams-Bashforth's method[7] by recalling eqs. (3.7) and (3.9), we find

$$\begin{cases} f(t)=\theta_m+a_1 t+a_2 t^2+\cdots\cdots, & (3.11)\\ s(t)=\qquad b_1 t+b_2 t^2+\cdots\cdots, & (3.13)\end{cases}$$

with

$$a_1=-\frac{20\lambda}{3cL^2}\theta_m,\qquad b_1=\frac{1}{\rho l}\left(Q(0)-\frac{cL}{3}a_1\right),$$
$$a_2=-\frac{5}{6\rho l}\left[-\frac{2c}{L}\theta_m a_1 b_1+ca_1^2+\frac{3}{cL}\left\{\frac{3}{cL}\theta_m b_1-ca_1\right)\left(\frac{cL}{5}a_1+\frac{4\lambda}{3L}\theta_m\right)+\rho l\left(-\frac{2c}{5}b_1+\frac{4\lambda}{3L}\right)a_1\right\}\Bigg]$$
$$a_3=\frac{1}{6\rho l}\left(\frac{3}{cL}-\frac{c}{L^2}\theta_m b_1^2+\frac{6}{L}a_1 b_1-2a_2\right)$$

etc.

§ 4. Conclusion

In the previous works,[8),9)] we investigated the melting elastic solid by Biot's variational method. After formulating the variational principle, we used the quadratic approximate formula as the test function. The method introduced in this paper is able to find the solution of the type presented as eq. (2.5).

Yu and Vujanovic investigated the problem of fixed boundary (0, L), and found the variational principle[1), 2)]

$$\delta V = 0. \qquad \cdots\cdots \qquad (4.1)$$

But our problem is the moving boundary (s, L), and the variational principle is eq. (2.4).

The method in this paper has a posibiljty of treating the problems in curvilinear coordinate in two or three dimensions, which we shall investigate later.

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